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Quantum field theory and local contextual extensions

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Abstract. It is shown that quantum field theory admits at least one local and causal extension which gives the same statistical predictions. A general algebraic formalism for the study of such extensions is presented. Quantum field theory and the corresponding extensions are considered in the framework of the Haag-Kastler approach. Obstruction due to the Bell inequalities is overcome by the use of a generalized probability theory.

1. Introduction

The aim of this paper is to present a general algebraic formalism for the study of causal and local field theories that are compatible at the statistical level with quantum field theory.

The paper is a logical continuation of an earlier study [6] where an algebraic formalism for contextual hidden variables, based on classical statistics, was presented.

In the present paper, we shall deal with local hidden variables within the algebraic quantum field theory scheme.

The following are the most important relations between [6] and the present study.

(i) *Contextuality and local contextuality.* As in [6], we deal here with contextual hidden variables. Let us recall that contextuality allows us to overcome obstructions which are implied by the classical works of von Neumann [15], Gleason [9], Bell [2] and others (see [3]). The obstructions mentioned are directly related to the problem of a consistent definition of subquantum states.

In a contextual hidden variable theory the value of a given quantum observable in a given subquantum state also depends on the *measurement context* [19]. Following [6], we shall interpret the contextuality as a consequence of inadequacy of the algebra of quantum observables for the complete description of physical reality.

However, in this paper we are interested in *local* theories only. This means that only a specific form of contextuality, the *local contextuality*, is admissible. The local contextuality forbids any correlation between results of quantum measurements performed in mutually causally (space-like) separated regions of space-time.

(ii) *Classical and non-classical probability concepts.* In any local and causal theory, classical (=Kolmogorovian) probability distributions satisfy Bell's inequalities [1]. On the other hand, Bell's inequalities are violated in quantum mechanics. Therefore, each hidden variable theory based on classical statistics and statistically compatible with quantum mechanics must be non-local.

However, the situation changes essentially if we generalize the concept of probability. In the framework of a (suitable) generalized statistics a unification of quantum mechanics, locality and causality becomes possible [7, 10–12, 16, 17]. Having this in mind one can justify the point of view according to which Bell's inequalities are not directly connected with locality, but rather with the type of statistics allowed. In implicit form, this statement is contained in Fine's theorem [8] too.

So far with quantum mechanics. A similar situation appears in algebraic quantum field theory as well. According to the work of Summers and Werner [20–22] Bell's inequalities are violated in this theory. In particular, Bell's inequalities are maximally violated in both Bose and Fermi free field theories in the vacuum state [21].

Consequently, the use of some kind of non-classical statistics in local causal refinements of algebraic quantum field theory (if such structures exist) is unavoidable.

In contrast to [6], the present paper is based on a 'contextual' modification of classical statistics. The paper is organized as follows. In section 2 we first describe the quantum structure. This includes basic axioms of quantum field theory as well as a representation for the measurement contexts. Quantum field theory will be treated in the framework of the algebraic approach of Haag and Kastler [13].

Concerning contexts, we shall distinguish two types: simple and composite. By definition, a simple context corresponds to a single (irreducible) measurement situation. On the other hand, composite contexts correspond to compositions of single measurements performed in causally separated regions of the spacetime.

Our next step is to formalize, in the framework of the Haag–Kastler approach, the idea of local contextual refinements. This will be done with the notion of local contextual extension. We then construct, starting from the quantum structure, an important C^* algebra, denoted by *Lctx*, and a local contextual extension which naturally corresponds to this algebra. We also show that the 'universality' of the algebra *Lctx*, with respect to an arbitrary local contextual extension, holds.

For our discussion, the most interesting local contextual extensions are those in which the statistics in the quantum states are interpretable as lack of knowledge about 'complete states'. Structures of this kind are investigated in section 3. The precise formulation leads to the notion of local hidden variable (LHV) extension. We establish the possibility LHVs by showing that the extension associated with the algebra *Lctx* is an LHV extension. We then analyse the most important general properties of LHV extensions. For a given LHV extension we introduce the concept of the complete (subquantum) state, of the subquantum space and of the 'classical' projection of the complete theory.

The mentioned modification of classical statistics, which is necessary to overcome the obstruction by Bell's inequalities, is contained only implicitly in the notion of an LHV extension. The analysis of the statistics of LHV extensions is the topic of section 4. We shall first introduce a general concept of 'contextual event space' and of a probability measure on it. We formulate an analogue of the weak law of large numbers for this 'contextual statistics', which ensures the possibility of relative-frequencies interpretation of probabilities. We then show that the subquantum space Ω of a given LHV extension naturally has the structure of a contextual event space and that each quantum state gives rise to a probability measure on it. The correspondence

$$\{\text{states on } \Sigma\} \to \{\text{probability measures on } \Omega\}$$

then allows us to interpret quantum stochasticity as a consequence of a lack of knowledge about hidden variables $\omega \in \Omega$.

Finally, we show that any representation D of the algebra Σ of quantum observables in a Hilbert space H naturally determines a 'projector valued measure' c_D on the subquantum space Ω. This entity contains all the information about the lack-of-knowledge interpretation of the quantum states in H. It can be considered as a 'quantum interpreter' of 'quantally actualizable' subquantum events.

In this paper, we shall deal with C^* algebras with unity and with unity-preserving homomorphisms. An ideal means a closed two-sided ideal.

2. Local contextual extensions

We start our analysis with a description of the quantum structure. Let $\mathfrak{U}$ be the family of all double cones in the Minkowski space-time M^4.

The principal entity of the 'quantum world', the C^* algebra of all quantum observables, will be denoted by Σ. We shall suppose that each $U \in \mathfrak{U}$ gives rise to a C^* subalgebra Σ_U of Σ, consisting of all observables 'localized' in U. We shall also suppose that there exists an action $\alpha: \Pi_+^\uparrow \ni g \to \alpha_g \in \mathrm{Aut}(\Sigma)$ of the connected component $\Pi_+^\uparrow$ of the Poincaré group, by automorphisms of Σ, such that the triplet $(\Sigma, \alpha, \{\Sigma_U;\, U \in \mathfrak{U}\})$ possesses the following properties:

(i) The algebra Σ is generated by $\{\Sigma_U;\, U \in \mathfrak{U}\}$; (1)

(ii) If $U \subseteq V$ then $\Sigma_U \subseteq \Sigma_V$;

(iii) For each $U \in \mathfrak{U}$ and $g \in \Pi_+^\uparrow$, one has $\alpha_g(\Sigma_U) = \Sigma_{g(U)}$;

(iv) If $U, V \in \mathfrak{U}$ are causally separated, then a linear map $F_{UV}: \Sigma_U \otimes_{\text{alg}} \Sigma_V \to \Sigma$ defined by $F_{UV}(\hat{a} \otimes \hat{b}) = \hat{a}\hat{b}$ is an injective $*$ homomorphism.

Here, $\otimes_{\text{alg}}$ denotes the algebraic tensor product. We shall now describe the representatives for simple and composite measurement contexts. We assume that each simple context is determined by a certain (non-trivial) commutative C^* subalgebra of Σ, which can be interpreted as the algebra of observables actualizable in this context. We shall denote by $\mathbf{T}$ the family of all simple contexts and, for each $U \in \mathfrak{U}$, by $\mathbf{T}_U$ the subfamily of $\mathbf{T}$ consisting of all simple contexts that can be realized in U.

We shall suppose that the following properties hold:

(i) $\mathbf{T} = \bigcup_{U \in \mathfrak{U}} \mathbf{T}_U$; (2)

(ii) For each $U \in \mathfrak{U}$, the family $\mathbf{T}_U$ generates Σ_U;

(iii) If $U \subseteq V$, then $\mathbf{T}_U \subseteq \mathbf{T}_V$;

(iv) For each $g \in \Pi_+^\uparrow$ and $U \in \mathfrak{U}$, one has $\alpha_g(\mathbf{T}_U) = \mathbf{T}_{g(U)}$.

To introduce composite contexts, we consider certain subsets of $\mathbf{T}$: by definition, a composite context is a finite set $A = \{A_1, \ldots, A_k\}$, where A_i are simple contexts and there are mutually causally separated sets $U_1, \ldots, U_k \in \mathfrak{U}$ such that $A_i \in T_{U_i}$. Let $\hat{\mathbf{T}}$ denote the family of all (simple and composite) contexts. The family $\mathbf{T}$ can be seen as a subfamily of $\hat{\mathbf{T}}$ (contexts of the form $\{A\}$).

For each context $A = \{A_1, \ldots, A_k\}$, we define its *domain* $\mathrm{dom}(A)$ to be the C^* subalgebra of Σ generated by $A_1, \ldots, A_k$. According to condition (iv) of (1), the algebra $\mathrm{dom}(A)$ is naturally isomorphic to $A_1 \otimes \ldots \otimes A_k$. We denote by $i_A: \mathrm{dom}(A) \to \Sigma$ the canonical inclusion and by Ω_A the spectrum of $\mathrm{dom}(A)$. The space Ω_A is naturally homeomorphic to $\Omega_{A_1} \times \ldots \times \Omega_{A_k}$.

The formula $g(A) = \{\alpha_g(A_1), \ldots, \alpha_g(A_k)\}$ defines a natural action of the group $\Pi_+^\uparrow$ on $\mathbf{T}$ and $\hat{\mathbf{T}}$. It is easy to see that $\mathrm{dom}[g(A)] = \alpha_g[\mathrm{dom}(A)]$, for each $g \in \Pi_+^\uparrow$ and $A \in \hat{\mathbf{T}}$.

In our exposition, the 'quantum world' will be represented by the quadruplet $(\Sigma, \alpha, \{\Sigma_U;\, U \in \mathfrak{U}\}, \mathbf{T})$.

Definition 2.1. A local contextual extension of $(\Sigma, \alpha, \{\Sigma_U;\, U \in \mathfrak{U}\}, \mathbf{T})$ is a triplet $(\Sigma', \phi, \{\iota_A;\, A \in \mathbf{T}\})$ with properties:

(i) Σ' is a C^* algebra and $\phi: \Sigma' \to \Sigma$ is a $*$ homomorphism;

(ii) The family $\{\iota_A;\, A \in \mathbf{T}\}$ consists of $*$ homomorphisms $\iota_A: A \to \Sigma'$ such that $i_A = \phi\iota_A$. The algebra Σ' is generated by the family of subalgebras $\{\iota_A(A);\, A \in \mathbf{T}\}$;

(iii) There exists an action $\alpha': \Pi^{\uparrow}_{+} \ni g \to \alpha'_g \in \mathrm{Aut}(\Sigma')$ of $\Pi^{\uparrow}_{+}$ by automorphisms of Σ' such that $\alpha'_g\iota_A = \iota_{g(A)}\alpha_g i_A$, for each $g \in \Pi^{\uparrow}_{+}$ and $A \in \mathbf{T}$;

(iv) For each $U \in \mathfrak{U}$, let Σ'_U denote the C^* subalgebra of Σ' generated by the family of algebras $\{\iota_A(A);\, A \in \mathbf{T}_U\}$. Then, if $U, V \in \mathfrak{U}$ are causally disconnected, a linear map $F'_{UV}: \Sigma'_U \otimes_{\mathrm{alg}} \Sigma'_V \to \Sigma'$ defined by $F'_{UV}(\hat{a}' \otimes \hat{b}') = \hat{a}'\hat{b}'$ is an injective $*$ homomorphism.

Remark. The properties (i) and (ii) say that the triplet $(\Sigma', \phi, \{\iota_A;\, A \in \mathbf{T}\})$ is a contextual extension of $(\Sigma, \mathbf{T})$, in the sense of [6].

In the following proposition we have collected the most important general properties of local contextual extensions.

Proposition 2.1. Let $(\Sigma', \phi, \{\iota_A;\, A \in \mathbf{T}\})$ be a local contextual extension of $(\Sigma, \alpha, \{\Sigma_U;\, U \in \mathfrak{U}\}, \mathbf{T})$. Then

(i) The equality $\alpha'_g\iota_A = \iota_{g(A)}\alpha_g i_A$ characterizes the action α';

(ii) The triplet $(\Sigma', \alpha', \{\Sigma'_U;\, U \in \mathfrak{U}\})$ satisfies conditions (1);

(iii) For each $\{A_1, \ldots, A_k\} = A \in \hat{\mathbf{T}}$ there exists the unique $*$ homomorphism $\iota_A: \mathrm{dom}(A) \to \Sigma'$ which extends the maps $\iota_{A_1}, \ldots, \iota_{A_k}$. This homomorphism is injective and satisfies the equations $\alpha'_g\iota_A = \iota_{g(A)}\alpha_g i_A$ and $\phi\iota_A = i_A$;

(iv) The homomorphism $\phi: \Sigma' \to \Sigma$ is surjective. For each $U \in \mathfrak{U}$ one has $\phi(\Sigma'_U) = \Sigma_U$. For each $g \in \Pi^{\uparrow}_{+}$ one has $\phi\alpha'_g = \alpha_g\phi$.

Proof. (i) The equality $\alpha'_g\iota_A = \iota_{g(A)}\alpha_g i_A$ determines the action α' on elements of the set $\{\iota_A(\hat{a});\, A \in \mathbf{T}, \hat{a} \in A\}$. On the other hand, the algebra Σ' is generated by these elements.

(ii) It is a consequence of the definition of the algebras Σ'_U, of the monotonicity of the correspondence $U \to \mathbf{T}_U$, of the fact that the family $\{\iota_A(A);\, A \in \mathbf{T}\}$ generates Σ', of the Poincaré-covariance of this family and of property (iv) at (1) for maps F'_{UV}.

(iii) The formula $\iota_A(\hat{a}_1 \otimes \ldots \otimes \hat{a}_k) = \iota_{A_1}(\hat{a}_1) \ldots \iota_{A_k}(\hat{a}_k)$ defines, for each $\{A_1, \ldots, A_k\} = A \in \hat{\mathbf{T}}$, a $*$ homomorphism $\iota_A: A_1 \otimes \ldots \otimes A_k = \mathrm{dom}(A) \to \Sigma'$. This homomorphism satisfies equations of (iii) in the above proposition. The second equality implies injectivity of ι_A. The uniqueness is a consequence of the fact that $\mathrm{dom}(A)$ is generated by $A_1, \ldots, A_k$.

(iv) The surjectivity of ϕ as well as equations $\phi(\Sigma'_U) = \Sigma_U$ are consequences of the fact that an image of a C^* homomorphism is closed [5] and of the fact that $\phi(\Sigma'_U)$ and $\phi(\Sigma')$ are dense in Σ_U and Σ respectively. The elements of the form $\hat{a}' = \iota_A(\hat{a})$ satisfy the equation $\alpha_g\phi(\hat{a}') = \phi\alpha'_g(\hat{a}')$. The set S of all elements of Σ' satisfying this equation is a C^* subalgebra of Σ'. On the other hand, S is dense in Σ'. Thus, it coincides with Σ'. □

A trivial example of a local contextual extension is the triplet $(\Sigma, id, \{i_A;\, A \in \mathbf{T}\})$. This is the 'smallest' one. A non-trivial example is the 'biggest' extension, the construction of which will now be given.

Let $Lctx(\Sigma, \{\Sigma_U; U \in \mathfrak{U}\}, \mathbf{T})$ be the C^* algebra generated by the set of symbols $\{(\hat{a}, A); A \in \mathbf{T}, \hat{a} \in A\}$ and the following relations:

$$\begin{aligned} &(1, A) = 1 \\ &(\hat{a}, A)(\hat{b}, A) = (\hat{a}\hat{b}, A) \\ &\alpha(\hat{a}, A) + \beta(\hat{b}, A) = (\alpha\hat{a} + \beta\hat{b}, A) \qquad \alpha, \beta \in C \\ &(\hat{a}, A)^* = (\hat{a}^*, A) \\ &(\hat{a}, A)(\hat{b}, B) = (\hat{b}, B)(\hat{a}, A) \qquad \text{if } \{A, B\} \in \hat{\mathbf{T}}. \end{aligned} \tag{3}$$

It is easy to see that for each $A \in \mathbf{T}$, the correspondence $\hat{a} \to (\hat{a}, A)$ defines a $*$ homomorphism $\hat{\iota}_A : A \to Lctx(\Sigma, \{\Sigma_U; U \in \mathfrak{U}\}, \mathbf{T})$. To simplify the notation, the algebra $Lctx(\Sigma, \{\Sigma_U; U \in \mathfrak{U}\}, \mathbf{T})$ will also be denoted by $Lctx$.

Proposition 2.2. Let Σ' be a C^* algebra and $\{\lambda_A; A \in \mathbf{T}\}$ a family of $*$ homomorphisms $\lambda_A : A \to \Sigma'$ such that $\lambda_A(\hat{a})\lambda_B(\hat{b}) = \lambda_B(\hat{b})\lambda_A(\hat{a})$ whenever $\{A, B\} \in \hat{\mathbf{T}}$. Then, there exists one and only one $*$ homomorphism $\lambda' : Lctx \to \Sigma'$ such that $\lambda'\hat{\iota}_A = \lambda_A$, for each $A \in \mathbf{T}$.

Proof. The algebra $Lctx$ is generated by the elements $\hat{\iota}_A(\hat{a})$. This implies the uniqueness of λ'. The existence follows from the facts that $\lambda_A(\hat{a})$ satisfy relations (3) and that each C^* algebra can be isometrically represented in some Hilbert space [5]. □

There are two examples in which this proposition can be naturally applied.

Example 1. Let $\Sigma' = \Sigma$ and $\lambda_A = i_A$. Then, there exists the unique $*$ homomorphism $\hat{\phi} : Lctx \to \Sigma$ such that $\hat{\phi}\hat{\iota}_A = i_A$.

Example 2. Let $\Sigma' = C$ and, for each $A \in \mathbf{T}$, let $\lambda_A : A \to C$ be an arbitrary character (non-trivial multiplicative linear $*$ functional) of A. Proposition 2.2 implies that there is the unique character $\lambda' : Lctx \to C$ such that $\lambda'\hat{\iota}_A = \lambda_A$. Conversely, any character λ' on $Lctx$, by the same formula, determines characters λ_A. Thus, the set of all characters of $Lctx$ can be identified with the Cartesian product $\Pi_{\mathbf{T}} = \prod_{A \in \mathbf{T}} \Omega_A$ of all spectra Ω_A.

The algebra $Lctx(\Sigma, \{\Sigma_U; U \in \mathfrak{U}\}, \mathbf{T})$ possesses a natural quasilocal structure, such that conditions (1) are satisfied. For $U \in \mathfrak{U}$, let $Lctx_U$ be a C^* subalgebra of $Lctx$ generated by the elements $\{\hat{\iota}_A(\hat{a}); A \in \mathbf{T}_U\}$. It is clear that $U \subseteq V$ implies $Lctx_U \subseteq Lctx_V$ and that the family $\{Lctx_U; U \in \mathfrak{U}\}$ generates $Lctx$. If $U, V \in \mathfrak{U}$ are causally separated then the elements of $Lctx_U$ commute with the elements of $Lctx_V$. Therefore, the map $\hat{F}_{UV} : Lctx_U \otimes_{\text{alg}} Lctx_V \to Lctx$ defined by $\hat{F}_{UV}(\hat{a} \otimes \hat{b}) = \hat{a}\hat{b}$ is a $*$ homomorphism. To show the injectivity of this map we apply proposition 2.2 to the following situation: $\Sigma' = Lctx_U \otimes_\alpha Lctx_V$, $\lambda_A = \hat{\iota}_A \otimes I$ for $A \in \mathbf{T}_U$, $\lambda_A = I \otimes \hat{\iota}_A$ for $A \in \mathbf{T}_V$ and $\lambda_A = f_A 1 \otimes 1$ otherwise, where f_A is an arbitrary character of A. The corresponding map $\lambda' : Lctx \to Lctx_U \otimes_\alpha Lctx_V$ satisfies the equation $\lambda'\hat{F}_{UV}(\hat{a} \otimes \hat{b}) = \lambda'(\hat{a})\lambda'(\hat{b}) = \hat{a} \otimes \hat{b}$. Thus, $\hat{F}_{UV}$ is invertible.

Finally, we introduce an action $\hat{\alpha}$ of $\Pi_+^\uparrow$ on $Lctx$. For $g \in \Pi_+^\uparrow$ we define $\lambda_A = \hat{\iota}_{g(A)}\alpha_g i_A$. According to proposition 2.2 there exists a unique $*$ homomorphism $\hat{\alpha}_g : Lctx \to Lctx$ such that $\hat{\alpha}_g\hat{\iota}_A = \lambda_A = \hat{\iota}_{g(A)}\alpha_g i_A$. It is easy to see that $g \to \hat{\alpha}_g$ is a representation of $\Pi_+^\uparrow$ by automorphisms of $Lctx$ and that $\hat{\alpha}_g(Lctx_U) = Lctx_{g(U)}$.

Our analysis of the algebra $Lctx$ is summarized in the following.

Proposition 2.3. (i) The triplet $(Lctx, \hat{\phi}, \{\hat{\iota}_A;\ A \in \mathbf{T}\})$ is a local contextual extension of $(\Sigma, \alpha, \{\Sigma_U;\ U \in \mathfrak{U}\}, \mathbf{T})$;

(ii) If $(\Sigma', \phi, \{\iota_A;\ A \in \mathbf{T}\})$ is a local contextual extension of $(\Sigma, \alpha, \{\Sigma_U;\ U \in \mathfrak{U}\}, \mathbf{T})$ then there exists one and only one * homomorphism $\lambda': Lctx \to \Sigma'$ such that $\lambda' \hat{\iota}_A = \iota_A$. This map is surjective, and has the following properties:

$$\hat{\phi} = \phi\lambda'$$

$$\lambda'(Lctx_U) = \Sigma'_U \qquad \text{for each } U \in \mathfrak{U}$$

$$\lambda' \hat{\alpha}_g = \alpha'_g \lambda' \qquad \text{for each } g \in \Pi^{\uparrow}_{+}.$$

□

3. Local contextual hidden variables

The notion of a 'local contextual hidden variable theory' is a symbiosis of two ideas: that of the local contextual refinement and of the lack-of-knowledge interpretation of the quantum stochasticity. We shall now formulate two important properties that characterize those local contextual extensions that are the base for an LHV theory.

Let $(\Sigma', \phi, \{\iota_A;\ A \in \hat{\mathbf{T}}\})$ be a local contextual extension and let us denote by $k(\Sigma')$ the minimal ideal in the algebra Σ' which contains all commutators.

Property (i). Statistical interpretation of quantum states. For any state ρ on Σ and finite subset $F \subseteq \mathbf{T}$ there exists a continuous linear functional ρ'_F on Σ' such that $\rho'_F|_{k(\Sigma')} = 0$ and $\rho'_F[\iota_A(\hat{a})] = \rho(\hat{a})$, for each context $A \subseteq F$ and $\hat{a} \in \mathrm{dom}(A)$.

Comparing this with definition 3.1 of [6], such a formulation may seem strange (we allow the F-dependence and we do not require the positivity of functionals ρ'_F). However, this formulation still ensures the possibility of the statistical foundation of quantum states, but in the framework of a 'contextual' statistics. This will be explained in detail in section 4.

Property (i) implies that $k(\Sigma') \neq \Sigma'$. Equivalently, the set of characters of Σ' is non-empty. Let us denote this set by Ω. The set Ω, endowed with the *-weak topology, becomes a compact topological space. The * homomorphism $e: \Sigma' \to C(\Omega)$, introduced by the formula $e(\hat{a}')(\omega) = \omega(\hat{a}')$, where $C(\Omega)$ is the C^* algebra of complex valued continuous functions on Ω, is surjective and $\ker(e) = k(\Sigma')$. The algebra $C(\Omega)$ is isometrically isomorphic to $\Sigma'/k(\Sigma')$.

From the point of view of hidden variables, the elements of the space Ω correspond to complete (subquantum) states of the field (complete field configurations). The number $\omega(\hat{a}')$ can be interpreted as the value of the variable $\hat{a}' \in \Sigma'$ in the state $\omega \in \Omega$.

The elements of Ω can also be characterized as states on Σ' having the zeroth dispersion on every element $\iota_A(\hat{a})$, where $A \in \mathbf{T}$ and $\hat{a}^+ = \hat{a} \in A$.

For each $U \in \mathfrak{U}$, the characters of the algebra Σ'_U can be interpreted as 'portions' of complete field configurations over U.

Property (ii). Extendibility of pairs of 'portions' over causally separated regions. For each $U, V \in \mathfrak{U}$ that are causally separated, and each pair of characters ω_1 on Σ'_U and ω_2 on Σ'_V, there exists at least one character $\omega \in \Omega$ such that $\omega|_{\Sigma'_U} = \omega_1$ and $\omega|_{\Sigma'_V} = \omega_2$.

Definition 3.1. The triplet $(\Sigma', \phi, \{\iota_A;\ A \in \mathbf{T}\})$ is called a *local hidden variables (LHV) extension* if it satisfies properties (i) and (ii).

The following proposition establishes the existence of LHV extensions.

Proposition 3.1. The triplet $(Lctx, \hat{\phi}, \{\hat{\iota}_A;\ A \in \mathbf{T}\})$ is an LHV extension.

Proof. The space $\Pi_{\mathbf{T}}$ of all characters of $Lctx$ is isomorphic to the product $\prod_{A \in \mathbf{T}} \Omega_A$. On the other hand, for each $U \in \mathfrak{U}$, the space $\Pi_{U,\mathbf{T}}$ of characters of $Lctx_U$ is isomorphic to the product $\prod_{A \in \mathbf{T}_U} \Omega_A$. If U, $V \in \mathfrak{U}$ are causally separated, then $\mathbf{T}_U \cap \mathbf{T}_V = \emptyset$, and we conclude that $\Pi_{\mathbf{T}}$ is of the form $\Pi_{U,\mathbf{T}} \times \Pi_{V,\mathbf{T}} \times \Pi'$. This implies property (ii).

To show that property (i) is fulfilled, we consider the maps $\{\hat{F}_A : \mathrm{dom}(A) \to C(\Pi_{\mathbf{T}});\ A \in \hat{\mathbf{T}}\}$ defined by $\hat{F}_A(\hat{a})(\omega) = \hat{\pi}_A(\omega)(\hat{a})$, where $\hat{\pi}_A : \Pi_{\mathbf{T}} \to \Omega_{A_1} \times \ldots \times \Omega_{A_k} = \Omega_A$ is the natural projection and $A = \{A_1, \ldots, A_k\}$. For each state ρ on Σ and a finite set $F \subseteq \mathbf{T}$, there exists a Hermitian continuous linear functional $\bar{\rho}_F$ on $C(\Pi_{\mathbf{T}})$ such that $\bar{\rho}_F[\hat{F}_A(\hat{a})] = \rho(\hat{a})$, for each context $A \subseteq F$ and $\hat{a} \in \mathrm{dom}(A)$. The functional ρ'_F on $Lctx$ obtained as 'pull back' of $\bar{\rho}_F$ satisfies the conditions of property (i). □

Let $(\Sigma', \phi, \{\iota_A;\ A \in \hat{\mathbf{T}}\})$ be an LHV extension. For each $A \in \hat{\mathbf{T}}$, we define a map $F_A : \mathrm{dom}(A) \to C(\Omega)$ to be the composition $F_A = e\iota_A$.

Proposition 3.2. For each finite set $\{A^1, \ldots, A^k\}$ of *composite* contexts, a relation $F_{A^1}(\hat{a}_1) + \ldots + F_{A^k}(\hat{a}_k) = 0$ implies the relation $\hat{a}_1 + \ldots + \hat{a}_k = 0$. In particular,

(i) The maps F_A are injective $*$ homomorphisms;

(ii) If $F_A(\hat{a}) = F_B(\hat{b})$, then $\hat{a} = \hat{b}$.

Proof. Let $F = A^1 \cup \ldots \cup A^k$ and suppose that $F_{A^1}(\hat{a}_1) + \ldots + F_{A^k}(\hat{a}_k) = 0$. This means $\iota_{A^1}(\hat{a}_1) + \ldots + \iota_{A^k}(\hat{a}_k) \in k(\Sigma')$. According to definition 3.1, for each state ρ on Σ there exists a state ρ' on Σ' such that $\rho'[\iota_{A^i}(\hat{a}_i)] = \rho(\hat{a}_i)$ and $\rho'[\iota_{A^1}(\hat{a}_1) + \ldots + \iota_{A^k}(\hat{a}_k)] = 0$. Therefore, $\rho[a_1 + \ldots + a_k] = 0$, for any state ρ on Σ. Thus, $\hat{a}_1 + \ldots + \hat{a}_k = 0$. □

The following proposition shows that every LHV extension naturally gives rise to a classical field theory. The proof is simple, and we omit it. For $U \in \mathfrak{U}$ let Ω_U be the set of characters of Σ'_U endowed with the $*$-weak topology. The space Ω_U is compact and there exists a natural continuous map (restriction of characters) $r_U : \Omega \to \Omega_U$. According to definition 3.1 (ii), r_U are surjective.

Let us consider the net $\{C_U = C(\Omega_U);\ U \in \mathfrak{U}\}$ of commutative C^* algebras. For each $U \in \mathfrak{U}$, the map $i_U : C_U \to C(\Omega)$, defined by $i_U(f)(\omega) = f(r_U(\omega))$, is an isometrical embedding. For this reason, we can speak of C_U as certain subalgebras of $C(\Omega)$.

Proposition 3.3. (i) The family of algebras $\{C_U;\ U \in \mathfrak{U}\}$ generates $C(\Omega)$;

(ii) $U \subseteq V$ implies $C_U \subseteq C_V$;

(iii) There exists one and only one action α'' of $\Pi_+^{\uparrow}$ by automorphisms of $C(\Omega)$ with the property $\alpha''_g e = e\alpha'_g$. For each $U \in \mathfrak{U}$ we have $\alpha''_g(C_U) = C_{g(U)}$;

(iv) For causally separated U, $V \in \mathfrak{U}$, the map $f_{UV} : C_U \otimes C_V \to C(\Omega)$ defined by $f_{UV}(a \otimes b) = ab$ is an injective $*$ homomorphism. □

In other words, $(C_U, \alpha'', \{C_U;\ U \in \mathfrak{U}\})$ satisfies properties (1) of the algebraic field theory. It is natural to interpret this field theory as a *classical face* of a field theory based on $(\Sigma', \alpha', \{\Sigma'_U;\ U \in \mathfrak{U}\})$ because it is obtained by 'forgetting' the non-commutativity of Σ'.

On the other hand, the quantum theory based on $(\Sigma, \alpha, \{\Sigma_U;\ U \in \mathfrak{U}\})$ is obtained by 'forgetting' contextuality. Therefore, the triplet $(\Sigma', \alpha', \{\Sigma'_U;\ U \in \mathfrak{U}\})$ can be considered as a *common refinement* for the quantum field theory and for the corresponding classical theory.

According to proposition 3.2, each simple context $A \in \mathbf{T}$ is isomorphic to $F_A(A)$. Let $\mathbf{T}^\Omega$ denote the collection of algebras $\{F_A(A);\ A \in \mathbf{T}\}$ and for each $U \in \mathfrak{U}$ let $\mathbf{T}_U^\Omega = \{F_A(A);\ A \in \mathbf{T}_U\}$. It is easy to see that the families $\mathbf{T}^\Omega$ and $\{\mathbf{T}_U^\Omega;\ U \in \mathfrak{U}\}$ satisfy properties (2).

Proposition 3.4. The algebra $Lctx(C(\Omega), \{C_U;\ U \in \mathfrak{U}\}, \mathbf{T}^\Omega)$ is naturally isomorphic to $Lctx(\Sigma, \{\Sigma_U;\ U \in \mathfrak{U}\}, \mathbf{T})$.

Proof. It is sufficient to prove that the correspondence $A \to F_A(A)$ induces bijections $\mathbf{T} \leftrightarrow \mathbf{T}^\Omega$ and $\mathbf{T}_U \leftrightarrow \mathbf{T}_U^\Omega$. Let us suppose that $F_A(A) = F_B(B)$ for some $A, B \in \mathbf{T}$. This means that for each $\hat{a} \in A$ there exists a $\hat{b} \in B$ such that $F_A(\hat{a}) = F_B(\hat{b})$ and vice versa. Proposition 3.2 implies $\hat{a} = \hat{b}$ and we conclude that $A = B$. □

4. Generalized concept of probability and LHV extensions

4.1. Contextual event spaces

In this subsection we shall present, independently of LHV extensions, a 'contextual' generalization of classical probability theory, free of problems with Bell's inequalities. In the next subsection we shall relate this contextual statistics with LHV extensions.

Let $\mathbf{K}$ be a simplicial complex and let us suppose that for each point $p \in \mathbf{K}^0$ a set Ω_p with a σ-field B_p of its subsets is given. For each simplex $\{p_1, \ldots, p_n\} = S \in \mathbf{K}$ let us define a set $\Omega_S = \Omega_{p_1} \times \ldots \times \Omega_{p_n}$ with a σ-field $B_S = B_{p_1} \times \ldots \times B_{p_n}$ (the σ-field generated by products of sets from B_{p_i}).

Definition 4.1. A *contextual event space of the type* $\{\Omega_S;\ S \in \mathbf{K}\}$ is a pair $(\Omega, \{\pi_S;\ S \in \mathbf{K}\})$ where Ω is a set and $\{\pi_S;\ S \in \mathbf{K}\}$ is a collection of surjective maps $\pi_S: \Omega \to \Omega_S$ such that $\pi_S(\omega) = (\pi_{p_1}(\omega), \ldots, \pi_{p_n}(\omega))$, for each $\omega \in \Omega$ and $S = \{p_1, \ldots, p_n\} \in \mathbf{K}$.

Let $(\Omega, \{\pi_S;\ S \in \mathbf{K}\})$ be a contextual event space. For each $S \in \mathbf{K}$, let us introduce a family $\mathbf{P}_S = \pi_S^{-1}(B_S)$ of subsets of Ω. Clearly, $\mathbf{P}_S$ is a σ-field on Ω isomorphic to B_S. Finally, let $\mathbf{P} = \bigcup_{S \in \mathbf{K}} \mathbf{P}_S$ and let $\hat{\mathbf{P}}$ be a σ-field generated by $\mathbf{P}$.

The motivation for such a definition comes from a possible 'contextual' interpretation. The points $p \in \mathbf{K}^0$ correspond to *simple experimental arrangements (simple contexts)*, the other simplexes $S \in \mathbf{K}$ correspond to *composite contexts*, the points $\omega \in \Omega$ correspond to *elementary events*, the elements of $\mathbf{P}_S$ are *events that can be actualized in the context* S and finally, the elements of $\mathbf{P}$ are actualizable events.

Definition 4.2. A *probability measure* on $(\Omega, \{\pi_S;\ S \in \mathbf{K}\})$ is a map $\mu: \mathbf{P} \to [0, 1]$ with the following properties:

(i) $\mu(\Omega) = 1$;

(ii) For each *finite* subset $F \subseteq \mathbf{K}^0$ there exists a real-valued measure μ_F on $\hat{\mathbf{P}}$ such that $\mu_F(\Lambda) = \mu(\Lambda)$, for each simplex $S \subseteq F$ and each $\Lambda \in \mathbf{P}_S$.

An immediate consequence of this definition is that restriction of μ to each $\mathbf{P}_S$ gives an ordinary probability measure.

For given contextual event spaces $\{(\Omega_i, \{\pi^i_S; S \in \mathbf{K}\}); i = 1, \ldots, k\}$ of the types $\{\Omega^i_S; S \in \mathbf{K}\}$ respectively, we can naturally define their *product*. To do this, let us consider the set $\Omega^* = \Omega_1 \times \ldots \times \Omega_k$ and maps $\pi^*_S = \pi^1_S \times \ldots \times \pi^k_S : \Omega^* \to \Omega^1_S \times \ldots \times \Omega^k_S$. It is easy to see that $(\Omega^*, \{\pi^*_S; S \in \mathbf{K}\})$ is a contextual event space of the type $\{\Omega^1_S \times \ldots \times \Omega^k_S; S \in \mathbf{K}\}$. Now, let

$$\mathbf{P}^i_S = (\pi^i_S)^{-1}(B^i_S) \qquad \mathbf{P}^i = \bigcup_{S \in \mathbf{K}} \mathbf{P}^i_S \qquad \mathbf{P}^*_S = \pi_S^{-1}(B^1_S \times \ldots \times B^k_S) \qquad \mathbf{P}^* = \bigcup_{i=1}^{k} \mathbf{P}^*_S.$$

We denote the σ-fields on Ω_i and Ω^*, that are generated by $\mathbf{P}^i$ and $\mathbf{P}^*$ respectively, by $\hat{\mathbf{P}}^i$ and $\hat{\mathbf{P}}^*$. The following proposition is a direct consequence of our definitions.

Proposition 4.1. (i) $\mathbf{P}^*_S = \mathbf{P}^1_S \times \ldots \times \mathbf{P}^k_S$ for each $S \in \mathbf{K}$;

(ii) $\hat{\mathbf{P}}^* \subseteq \hat{\mathbf{P}}^1 \times \ldots \times \hat{\mathbf{P}}^k$;

(iii) If μ_i are probability measures on $(\Omega_i, \{\pi^i_S; S \in \mathbf{K}\})$ respectively, then there exists a unique probability measure $\mu^* = \mu_1 \times \ldots \times \mu_k$ on $(\Omega^*, \{\pi^*_S; S \in \mathbf{K}\})$ such that

$$\mu^*(\Lambda_1 \times \ldots \times \Lambda_k) = \mu_1(\Lambda_1) \ldots \mu_k(\Lambda_k)$$

for each $S \in \mathbf{K}$ and $\Lambda_i \in \mathbf{P}^i_S$. □

For our discussion, it is very important to consider the case $\Omega_1 = \ldots = \Omega_k$ and $\mu_1 = \ldots = \mu_k = \mu$. This corresponds to the situations in which the initial experiments are repeated k-times. The following proposition then establishes a 'contextual version' of Bernoulli's law of large numbers. The proof can be easily reduced to the proof of the ordinary Bernoulli law. For $\omega^* \in \Omega^*$ and $\Lambda \in \mathbf{P}$, let $r_\Lambda(\omega^*)$ denote the relative frequency of occurrence Λ in ω^*.

Proposition 4.2. For each $\Lambda \in \mathbf{P}$ and $\varepsilon > 0$, let $\Lambda^*_\varepsilon \subseteq \Omega^*$ be the set of all $\omega^* \in \Omega^*$ in which $|r_\Lambda(\omega^*) - \mu(\Lambda)| \geqslant \varepsilon$. Then

(i) $\Lambda_\varepsilon \in \mathbf{P}^*$;

(ii) $\mu^*(\Lambda^*_\varepsilon) \leqslant 1/(4k\varepsilon^2)$. □

The fact that the law of large numbers holds implies (just as in the classical case) the possibility of empirical foundation of these statistics. All probabilities are interpretable in terms of relative frequencies of occurrence.

4.2. *Statistics of* LHV *extensions*

We now go back to the quantum field theory. The collection $\mathbf{T}$ of all simple contexts has a natural structure of a simplicial complex: simplexes are just elements of the family $\hat{\mathbf{T}}$. For each $A \in \hat{\mathbf{T}}$ let B_A denote the Baire σ-field of Ω_A.

Let $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ be an LHV extension and Ω the space of characters of Σ'. For each context $A \in \hat{\mathbf{T}}$ let $\pi_A : \Omega \to \Omega_A$ denote the map defined by $\pi_A(\omega) = \omega\iota_A$. This map is obtained by transposing the homomorphism $F_A : \mathrm{dom}(A) \to C(\Omega)$. Proposition 3.2(i) implies that π_A is continuous and surjective. In terms of the identification $\Omega_A = \Omega_{A_1} \times \ldots \times \Omega_{A_k}$, the equality $\pi_A(\omega) = (\pi_{A_1}(\omega), \ldots, \pi_{A_k}(\omega))$ holds. Consequently:

Proposition 4.3. The pair $(\Omega, \{\pi_A; A \in \hat{\mathbf{T}}\})$ is a contextual event space of the type $\{\Omega_A; A \in \hat{\mathbf{T}}\}$. □

For each $A \in \hat{\mathbf{T}}$, let $\mathbf{P}_A = \pi_A^{-1}(B_A)$ and $\mathbf{P} = \bigcup_{A \in \hat{\mathbf{T}}} \mathbf{P}_A$. It is instructive to think about the elements of $\mathbf{P}$ as of quantally interpretable subquantum events. The elements of $\mathbf{P}_A$ correspond to events that can be actualized in the context A.

Proposition 4.4. For each $g \in \Pi_+^\uparrow$ and $A \in \hat{\mathbf{T}}$ one has

$$d'(g)(\mathbf{P}_A) = \mathbf{P}_{g(A)}.$$

In particular, $\mathbf{P}$ is Poincaré-invariant. Here, d' is the induced action of $\Pi_+^\uparrow$ on Ω.

Proof. Definition 2.1 implies $\alpha'_g \iota_A = \iota_{g(A)} \alpha_g i_A$ for each $A \in \hat{\mathbf{T}}$ and $g \in \Pi_+^\uparrow$. At the level of the spaces of characters, this equality has the following form:

$$\pi_{g(A)} d'(g) = d_A(g) \pi_A \tag{4}$$

where $d_A(g): \Omega_A \to \Omega_{g(A)}$ is a homeomorphism defined by $d_A(g)(\omega) = \omega \alpha_g^{-1} i_{g(A)}$. The statement of the proposition is a direct consequence of this equality and of the definition of $\mathbf{P}$. □

Now we shall see that any quantum state ρ (a state on Σ) determines canonically a probability measure μ_ρ on $(\Omega, \{\pi_A;\, A \in \hat{\mathbf{T}}\})$. For each $A \in \hat{\mathbf{T}}$, let $\mu_{\rho,A}$ denote the probability measure on (Ω_A, B_A) induced by the state ρi_A.

Proposition 4.5. For each quantum state ρ there exists one and only one probability measure $\mu_\rho : \mathbf{P} \to [0, 1]$ on $(\Omega, \{\pi_A;\, A \in \hat{\mathbf{T}}\})$ such that $\mu_\rho(\Lambda) = \mu_{\rho,A}(\pi_A(\Lambda))$ for each $A \in \hat{\mathbf{T}}$ and $\Lambda \in \mathbf{P}_A$.

Proof. It is clear that the measure μ_ρ is, if it exists, unique. Let us show its existence. According to condition (i) of definition 3.1, for each finite set $F \subseteq \mathbf{T}$ and each state ρ on Σ there exists a continuous linear functional $\bar{\rho}$ on $C(\Omega)$ such that $\bar{\rho}(F_A(\hat{a})) = \rho(\hat{a})$ for each $A \subseteq F$ and $\hat{a} \in \mathrm{dom}(A)$. The functional $\bar{\rho}$ induces [14] a real-valued measure ν on the Borel σ-field $B(\Omega)$. For $\Lambda \in \mathbf{P}_A$, this measure satisfies the equality $\nu(\Lambda) = \mu_{\rho,A}(\pi_A(\Lambda))$. In particular, if

$$\Lambda \in \mathbf{P}_{A_1} \cap \mathbf{P}_{A_2} \qquad \text{then} \qquad \mu_{\rho,A_1}(\pi_{A_1}(\Lambda)) = \mu_{\rho,A_2}(\pi_{A_2}(\Lambda)).$$

Consequently, for $\Lambda \in \mathbf{P}_A$, the formula $\mu_\rho(\Lambda) = \mu_{\rho,A}(\pi_A(\Lambda))$ consistently defines a map $\mu_\rho : \mathbf{P} \to [0, 1]$ with the desired properties. □

It is natural to interpret the elements of the family $\mathbf{P}$ as *subquantum events that possess a quantum meaning.* Accordingly, there should exist a natural map ('quantum interpreter'):

$$\{\text{elements of } \mathbf{P}\} \to \{\text{quantum events}\}$$

compatible with the statistical averages. Our next proposition establishes the existence, uniqueness and the main properties of such an entity. Let D be a faithful representation of Σ in a Hilbert space H. The *quantum events* (corresponding to D) can then be identified with the projectors in the bicommutant $D(\Sigma)''$.

Proposition 4.6. (i) There exists one and only one projector valued map $c_D : \mathbf{P} \to P(H)$ such that $\mu_\rho(\Lambda) = \mathrm{Tr}(c_D(\Lambda)\hat{\rho})$ for each state ρ induced by a statistical operator $\hat{\rho}$ in H and each $\Lambda \in \mathbf{P}$;

(ii) For each $A \in \hat{\mathbf{T}}$, let $c_{A,D}: B_A \to P(H)$ denote the spectral measure of the representation Di_A of dom(A) in H. Then $c_D(\Lambda) = c_{A,D}(\pi_A(\Lambda))$, for each $\Lambda \in \mathbf{P}_A$. In particular, a restriction of c_D to $\mathbf{P}_A$ gives a projector valued measure;

(iii) The image of the map c_D is contained in the bicommutant of $D(\Sigma)$ in H;

(iv) Let $\Lambda_1, \ldots, \Lambda_k$ be mutually disjoint elements of $\mathbf{P}$ and $\Omega = \Lambda_1 \cup \ldots \cup \Lambda_k$. Then $c_D(\Lambda_1), \ldots, c_D(\Lambda_k)$ are mutually orthogonal projectors and $c_D(\Lambda_1) + \ldots + c_D(\Lambda_k) = I$. In particular, the restriction of c_D to any finite Boolean subalgebra of $\mathbf{P}$ is a projector valued measure.

Proof. For $\Lambda \in \mathbf{P}_A$ we define $c_D(\Lambda)$ to be equal to $c_{A,D}(\pi_A(\Lambda))$. A necessary and sufficient condition for the existence of a global map $c_D: \mathbf{P} \to P(H)$ is that $c_{A,D}(\pi_A(\Lambda))$ does not depend on A. Let $\Lambda \in \mathbf{P}_{A_1} \cap \mathbf{P}_{A_2}$ and $F = A_1 \cup A_2$. According to the same reasoning as in the previous proof, there exists, for each statistical operator $\hat{\rho}$ in H, a real-valued measure ν on $B(\Omega)$ such that $\nu(\Lambda) = \mathrm{Tr}[\hat{\rho} c_{A,D}(\pi_A(\Lambda))]$ for each context $A \subseteq F$ and $\Lambda \in \mathbf{P}$. Especially,

$$\mathrm{Tr}[\hat{\rho} c_{A_1,D}(\pi_{A_1}(\Lambda))] = \mathrm{Tr}[\hat{\rho} c_{A_2,D}(\pi_{A_2}(\Lambda))]$$

which implies the equality $c_{A_1,D}(\pi_{A_1}(\Lambda)) = c_{A_2,D}(\pi_{A_2}(\Lambda))$.

Let us now suppose that another map $c'_D: \mathbf{P} \to P(H)$ satisfying the equation $\mathrm{Tr}(\hat{\rho} c'_D(\Lambda)) = \mu_\rho(\Lambda)$ is given. Then $\mathrm{Tr}(\hat{\rho}(c_D(\Lambda) - c'_D(\Lambda))) = 0$, for each $\Lambda \in \mathbf{P}$ and each statistical operator $\hat{\rho}$ in H. Thus, $c_D(\Lambda) = c'_D(\Lambda)$.

The fact that the image of the map c_D is contained in the bicommutant of $D(\Sigma)$ is a consequence of the fact that the image of each spectral measure $c_{A,D}$ is contained [14] in the bicommutant of $D[\mathrm{dom}(A)]$.

Finally, let $\Lambda_1, \ldots, \Lambda_k \in \mathbf{P}$ form a decomposition of Ω. Then for each statistical operator $\hat{\rho}$ in H there exists a normed real-valued measure ν on $B(\Omega)$ such that $\nu(\Lambda_i) = \mathrm{Tr}(\hat{\rho} c_D(\Lambda_i))$, $i \in \{1, \ldots, k\}$. Summing over all i and taking into account the arbitrariness of $\hat{\rho}$ we obtain an equality $c_D(\Lambda_1) + \ldots + c_D(\Lambda_k) = I$. On the other hand, if a sum of projectors is equal to unity then they are mutually orthogonal. □

In this proposition, we have not assumed that the representation D is Poincaré-covariant. If this is the case, then the 'quantum interpreter' c_D is also Poincaré-covariant.

More precisely, let us suppose that a unitary representation $\Delta: \hat{\Pi}^{\uparrow}_{+} \to U(H)$ of the universal covering $\hat{\Pi}^{\uparrow}_{+}$ of $\Pi^{\uparrow}_{+}$ is given, such that $D(\alpha_g(\hat{a})) = \Delta(\hat{g}) D(\hat{a}) \Delta(\hat{g})^+$ for each $\hat{g} \in \hat{\Pi}^{\uparrow}_{+}$ and $\hat{a} \in \Sigma$. Here, $g \in \Pi^{\uparrow}_{+}$ is the projected $\hat{g} \in \hat{\Pi}^{\uparrow}_{+}$.

Proposition 4.7. For $\Lambda \in \mathbf{P}$ and $g \in \hat{\Pi}^{\uparrow}_{+}$, the following equality holds:

$$c_D[d'(g)(\Lambda)] = \Delta(\hat{g}) c_D(\Lambda) \Delta(\hat{g})^+.$$

Proof. It is easy to see that $c_{g(A),D}[d_A(g)(S)] = \Delta(\hat{g}) c_{A,D}(S) \Delta(\hat{g})^+$, for each $A \in \hat{\mathbf{T}}$, $S \in B_A$ and $\hat{g} \in \hat{\Pi}^{\uparrow}_{+}$. Together with the equality (4) and property (ii) of proposition 4.6, we obtain the desired equality. □

5. Concluding remarks

In this study we have shown that algebraic quantum field theory can be interpreted as a coarse-grained face of a local and causal subquantum theory so that, from the point of view of this finer theory, the whole quantum stochasticity is explained via the lack-of-knowledge scheme.

It is instructive to compare the presented subquantum theory with Rédei's study [18], according to which algebraic quantum field theory does not admit any 'local hidden variables' refinement.

For an appropriate quasilocal structure $(\Sigma, \{\Sigma_U;\, U \in \mathfrak{U}\})$, representing the 'quantum world', Rédei introduces a notion of local hidden theory, which consists of another quasilocal structure $(\Sigma', \{\Sigma'_U;\, U \in \mathfrak{U}\})$ and a unital positive linear map $L:\Sigma' \to \Sigma$ such that, besides appropriate locality and covariance conditions for L, each quantum state ρ is realizable as a probability measure μ_ρ on the state space $S(\Sigma')$ of Σ' in the sense that

$$\rho(L(\hat{b})) = \int_{S(\Sigma')} \omega(\hat{b})\, \mathrm{d}\mu_\rho(\omega) \tag{5}$$

for *each* $\hat{b} \in \Sigma'$.

The map L is a counterpart for the 'forgetting homomorphism' ϕ in the presented approach of LHV extensions.

In decomposition (5) it is also assumed that the dispersion of $\hat{b}$ in ω is, for each $\omega \in \mathrm{supp}(\mu_\rho)$, less than the dispersion of $L(\hat{b})$ in ρ (see [18] for details). Here, $\mathrm{supp}(\mu_\rho)$ is the support of μ_ρ.

The restriction to causal theories, which we are interested in, is equivalent to

$$\mathrm{supp}(\mu_\rho) \subseteq \Omega$$

for each ρ. Here, Ω is the space of characters (dispersion-free states) of Σ'.

Relation (5) now becomes

$$\rho(\phi(\hat{b})) = \int_{\Omega} \omega(\hat{b})\, \mathrm{d}\mu_\rho(\omega). \tag{6}$$

However, compared with property (i) in definition 3.1, condition (6) is much stronger. First of all, there is no physical reason for (6) to be valid for *all* $\hat{b}$, because it may happen that some $\hat{b}$ are not accessible through any quantum measurement. From the point of view of contextual extensions, only elements of the form $\hat{b} = \iota_A(\hat{a})$, where $A \in \hat{\mathbf{T}}$ and $\hat{a} \in \mathrm{dom}(A)$, should figure in (6).

Secondly, even if we require (6) to hold only for $\iota_A(\hat{a})$'s, the Bell inequalities obstruction still remains. This is a direct consequence of the results of Summers and Werner already mentioned, and the fact that μ_ρ in (6) *is* a classical (Kolmogorovian) probability measure.

On the other hand, the concept of contextual statistics, incorporated in the approach of LHV extensions, allows us to overcome the difficulties with Bell's inequalities.

Of course, contextual statistics is not the only way of overcoming the obstruction of the Bell inequalities. For example, statistics based on σ-classes [10, 11] or on amplitude densities [12] also have this property.

However, disregarding technical details, any other statistics should contain the contextual statistics as its part. This is so because the domain of the contextual statistics is exactly the family of quantally actualizable events. On the other hand, any satisfactory statistical description should include these events.

To conclude, let us relate the approach of LHV extensions, and Bell's 'beable quantum field theory' [4].

If $(\Sigma', \phi, \{\iota_A;\, A \in \mathbf{T}\})$ is an LHV extension, then the elements of the form $\{\iota_A(\hat{a});\, A \in \mathbf{T},\, \hat{a} = \hat{a}^+ \in A\}$ are interpretable as 'elementary beables' of the theory, because they possess a direct physical meaning (as quantum observables viewed through simple measurement contexts), and definite values in each subquantum state $\omega \in \Omega$.

Other Hermitian elements of Σ' are of the 'beable type', too. This is so because 'elementary beables' generate the whole Σ' (see definition 2.1, property (ii)), and because a state ρ' on Σ', dispersion-free on elementary beables is dispersion-free on the whole Σ' (that is, $\rho' \in \Omega$).

In this sense the theory based on LHV extensions is a special case of the beables quantum field theory of Bell, because all variables are –beables.

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